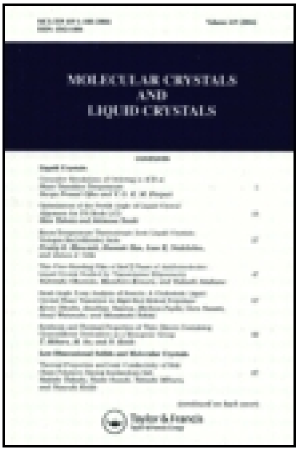

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LIGHT PROPAGATION IN STRATIFIED ANISOTROPIC MEDIA: ORTHOGONALITY AND SYMMETRY PROPERTIES OF THE 4X4 MATRIX FORMALISMS

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Abstract

For light propagation in stratified media the normal component of the Poynting vector is defined as an indefinite scalar product. The vanishing of this scalar product for two waves is regarded as their mutual orthogonality. Orthogonality in this sense is an inherent property of optical eigenmodes in lossless media. It is shown that the matrices D and P appearing in Berreman's 4x4 matrix formalism are hermitian and unitary, respectively, within this metric. The application of these properties can facilitate numerical calculations and analytical treatments.

INTRODUCTION

Light propagation in inhomogeneous anisotropic media, e.g. deformed liquid crystals, can be described by means of various 4x4 matrix formalisms[1-5]. According to the layer model a one dimensionally inhomogeneous medium is divided into a sufficiently large number of homogeneous layers. Inside the layers the wave equation can be solved easily. The solutions in adjacent layers can be connected using the boundary conditions for the tangential components of the electromagnetic field. Usually the direction of stratification, i.e. the layer normal, is chosen as z-axis. In Berreman's formalism[1,2] the four tangential components of the electromagnetic field are regarded as a 4-vector Ψ which is given by

$$\Psi^T = (\sqrt{\varepsilon_o}\, E_x,\ \sqrt{\mu_o}\, H_y,\ \sqrt{\varepsilon_o}\, E_y,\ -\sqrt{\mu_o}\, H_x) \quad , \qquad (1)$$

where ε_o and μ_o are the permittivity and permeability, resp., of the vacuum. The boundary conditions require that this vector at the output-side of the n'th layer coincides with that of the input-side of the n+1'st layer. The connection between input- and output-sides of the n'th layer can be written as

$$\Psi_n = P_n \Psi_{n-1}, \tag{2}$$

where P is the so-called propagator. Eq.(2) is the solution of the equation within the layer (the index n is omitted)

$$\frac{\partial}{\partial z} \Psi = i \frac{\omega}{c} D \Psi , \tag{3}$$

where D is given by $D =$

$$\begin{Bmatrix} \left(-k_x \frac{\varepsilon_{zx}}{\varepsilon_{zz}}\right) & \left(1 - \frac{k_x^2}{\varepsilon_{zz}}\right) & \left(-k_x \frac{\varepsilon_{zy}}{\varepsilon_{zz}}\right) & 0 \\ \left(\varepsilon_{xx} - \frac{\varepsilon_{xz}\varepsilon_{zx}}{\varepsilon_{zz}}\right) & \left(-k_x \frac{\varepsilon_{xz}}{\varepsilon_{zz}}\right) & \left(\varepsilon_{xy} - \frac{\varepsilon_{xz}\varepsilon_{zy}}{\varepsilon_{zz}}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\varepsilon_{yx} - \frac{\varepsilon_{yz}\varepsilon_{zy}}{\varepsilon_{zz}}\right) & \left(-k_x \frac{\varepsilon_{yz}}{\varepsilon_{zz}}\right) & \left(\varepsilon_{yy} - \frac{\varepsilon_{yz}\varepsilon_{zy}}{\varepsilon_{zz}} - k_x^2\right) & 0 \end{Bmatrix} \tag{4}$$

and $P = \exp\left(i\,(\omega/c)\,D\,d\right)$,

d is the thickness of the layer. Eq.(3) is a straightforward consequence of Maxwell's equations. However, because of its importance for the 4x4 matrix formalism it is often referred to as Berreman's equation. The repeated application of Eq.(2) and the boundary conditions leads to the general solution

$$\Psi_N = \left(\prod_{n=N}^{1} P_n\right) \Psi_o . \tag{5}$$

In homogeneous media, i.e. within each layer of the inhomogeneous medium, P and D have the same eigenvectors which are called optical eigenmodes (OEM's). Their eigenvalues correspond to phase factors and z-components of the wave vectors, resp.:

$$\begin{aligned} D\,\Psi_\alpha(z) &= k_{z\alpha}\,\Psi_\alpha(z), \\ P\,\Psi_\alpha(z+d) &= \exp\left\{ i\,(\omega/c)\,k_{z\alpha}\,d \right\}\,\Psi_\alpha(z). \end{aligned} \tag{6}$$

There are four OEM's Ψ_α ($\alpha = 1,..,4$) for a given P or D with the wave vectors

$$\mathbf{k}_\alpha = \frac{\omega}{c}\,(\,k_x,\,0\,,\,k_{z\alpha}\,). \tag{7}$$

Because of the reflection and refraction laws the tangential components of these wave vectors are equal within each layer of the medium. Without loss of generality can be assumed that $k_y = 0$. $k_x = n_o \sin\alpha_o$ can be easily determined when the refractive index n_o and the incidence angle α_o in the "outer" (with index o) medium (which is mostly assumed to be isotropic) is known.

SYMMETRY PROPERTIES

It has been known since the introduction of the 4x4 matrix method[1] that the matrices P and D contain less than 16 (actually 10) independent elements. For example the following relations are true for the elements $D_{\alpha\beta}$ of the matrix D

$$D_{11} = D^*_{22};\; D_{33} = D^*_{44};\; D_{41} = D^*_{23};\; D_{32} = D^*_{14};\; D_{42} = D^*_{13};\; D_{31} = D^*_{24}.$$

However, there has been no rigorous proof of the meaning of this symmetry. Such a proof is given in this section. We define the z-component of the crossed Poynting vector $S_{\alpha\beta}$ of two OEM (α,β)

$$S_{\alpha\beta} = E^*_{x\alpha}\,H_{y\beta} + E_{x\beta}\,H^*_{y\alpha} - E^*_{y\alpha}\,H_{x\beta} - E_{y\beta}\,H^*_{x\alpha}$$

as an indefinite scalarproduct $[\Psi_\alpha,\Psi_\beta]$ of the vectors Ψ_α and Ψ with

$$[\,\Psi_\alpha\,,\,\Psi_\beta\,] \equiv \frac{1}{c}\,S_{\alpha\beta} = \Psi^+_\alpha\;\Sigma_x\,\Psi_\beta\,, \tag{8}$$

where $\Sigma_x = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$ and $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c is the vacuum velocity of light and $^+$ denotes the transposed and conjugated vector). Indefinite means that the metric tensor Σ_x used therein has both positive and negative eigenvalues, therefore it is not positive definitely. However, the sign physically distinguishes between waves carrying energy in opposite directions with respect to the z-axis. With regard to this scalar product (8) the definition of hermiteicity

$$[\, D\,\Psi_\alpha \,,\, \Psi_\beta \,] \quad = \quad [\, \Psi_\alpha \,,\, D\,\Psi_\beta \,] \tag{9}$$

leads to the relations

$$(\, D\,\Psi_\alpha \,)^+\, \Sigma_x\, \Psi_\beta \quad = \quad \Psi_\alpha^+\, \Sigma_x\, (\, D\,\Psi_\beta \,) \tag{10}$$

and

$$D^+\, \Sigma_x \quad = \quad \Sigma_x\, D\,.$$

Using Eq.(3) the latter relation can be easily checked to be true. Hence, D is an hermitian matrix with respect to the definition of the scalar product. This symmetry is immediately connected with energy conservation[7]. Considering its spatial derivative and using Eq. (10) the scalar product (8) is found to be a constant

$$\partial/\partial z\ (\Psi^+\ \Sigma_x\ \Psi) = i\ (\omega/c)\ \Psi^+\ (\Sigma_x D - D^+\Sigma_x)\ \Psi = 0\,. \tag{11}$$

Keeping in mind that $S_{\alpha\beta}$ is the z-component of the Poynting vector Eq. (11) represents the energy conservation. Especially, for OEM with different real wave vectors we get

$$\partial/\partial z\ (\Psi_\alpha^+\ \Sigma_x\ \Psi_\beta) = i\ (\omega/c)\ (k_{z\alpha}^* - k_{z\beta})\ (\Psi_\alpha^+\ \Sigma_x\ \Psi_\beta) = 0 \tag{12}$$

instead of Eq.(11).

This means that

$$S_{\alpha\beta} = 0 \qquad\qquad \text{if } k_{z\alpha}^* \neq k_{z\beta}. \tag{13}$$

As a consequence of the definition of $S_{\alpha\beta}$ as a scalar product its vanishing for two waves (α,β) is regarded as their mutual orthogonality. Orthogonality in this sense is an inherent property of OEM with real and different wave vectors in lossless media where $\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}^*$. It includes the well-known cases of geometrically orthogonal polarization states in isotropic media as well as more general forms of

polarizations. For plane waves in birefringent media this is a consequence of the so-called Potier's relation[6,8].
In the same sense the matrix P can be shown to be unitary. Evidently, the definition of unitarity

$$[P \Psi_\alpha , P \Psi_\beta] = [\Psi_\alpha , \Psi_\beta] \quad (14)$$

is satisfied by P since the scalar product (8) is independent on z, i.e. the energy carried in z-direction is a constant, and a multiplication by P represents a shift in z-direction. Eq.(14) leads to the relations

$$(P \Psi_\alpha)^+ \Sigma_x (P \Psi_\beta) = \Psi_\alpha \Sigma_x \Psi_\beta \quad (15)$$

and

$$P^+ \Sigma_x P = \Sigma_x$$

which are very usefull when the inverse of P is needed:

$$P^{-1} = \Sigma_x P^+ \Sigma_x . \quad (16)$$

Hereby the inversion is reduced to a permutation of the rows and columns of the transposed and conjugated matrix. Since this relation is applicable both to a single layer and to the whole layer sequence it is not necessary to repeat all the matrix multiplications in opposite direction[5]. More applications of these symmetry properties can be found in[9].

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